A vertex model with an unbounded number of components

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Abstract. A model which is an infinite-component generalization of the six-vertex model is proposed. The method of solution is based on a random walk representation of states. The Bethe ansatz equations for periodic boundary conditions are obtained. The partition function of the finite model with special arrow configurations at the boundary of a square is given. In contrast to analogous solutions of Korepin and Baxter, it has the form of an evaluable determinant.

1. Introduction

Since the solution of the ice, F and KDP models by Lieb [1, 2], much attention has been given to generalizations of the six-vertex model. The most well known results obtained in this direction are Baxter's solution of the symmetric eight-vertex model and the hard-hexagon model [3]. The other line of generalization is an adaptation of the Bethe ansatz and related techniques to multicomponent systems on two-dimensional lattices. These include the Potts model [4], the $O(N)$ model [5], various non-intersecting string models [6-8], the $Z(n) \times Z(n)$ model of Belavin [9], the generalized six-vertex model [10]; see [11] for a review.

In spite of the variety of models solved to date, all of them have a common property: a finiteness of the number of components or vertex configurations. In this paper we pursue two aims. First, to present an exactly solvable model with an unbounded number of components. Second, to solve the model on a finite lattice of $N \times N$ sites with fixed boundary conditions. A similar problem arised firstly in the course of calculations of norms of Bethe wavefunctions for the quantum nonlinear Schrödinger equations [12] and for the quantum Heisenberg chain [13].

For two-dimensional classical models, this problem is equivalent to calculations of the partition function for a finite lattice. Korepin [14] obtained the partition function of the six-vertex model on a finite square with special boundary conditions. Properties of this solution were exploited by Izergin [15] to find the partition function in the form of a determinant. Baxter [16] suggested a new method called the perimeter Bethe ansatz which permits the finding of a solution of the Z-invariant six-vertex model on an arbitrary bounded lattice. In spite of the closed form, the partition functions obtained in [14, 15] and [16] are merely formal solutions as they contain sums of an increasing number of terms and some delicate limits need be taken in order to extract an explicit solution. The model suggested in this paper, with the boundary conditions used in [14], has a solution in the form of an evaluable determinant.

This paper is organized as follows. In section 2 the model is formulated and various boundary conditions are discussed. In section 3 the method of derivation of the Bethe ansatz equations [17] based on a random walk representation of the states of the model [18] is outlined. In section 4 this method is applied to the case of periodic boundary conditions for the lattice drawn diagonally. In section 5 we present the solution of the model on a finite square with Korepin's boundary conditions [14]. Section 5 is devoted to a short discussion.

2. The model

Firstly we consider the ordinary six-vertex model on a square lattice. It can be related to the two-component models: each bond of the lattice accepts one of two possible states. The states of bonds are denoted usually by arrows pointing left or right on horizontal bonds and up or down on vertical ones. The ice condition requires that there are precisely two arrows pointing into each vertex and two out.

In the general six-vertex model, each of the vertices has a distinct Boltzmann weight: $w_1, w_2, w_3, w_4, w_5, w_6$. The problem consists in determining the partition function

$$Z = \sum_{\{C\}} \prod_i w_{\xi(i)} \tag{1}$$

where the product is over all lattice sites and the sum is over all arrangements of the arrows allowed by the ice rule. The ice model will be obtained by setting $w_1 = w_2 = \ldots = w_6 = 1$.

Now, let us suppose that each horizontal bond contains either one arrow pointing left or an arbitrary number of arrows pointing right. The remaining vertical bonds remain are two state. We require that the numbers of right arrows on two adjacent horizontal bonds differ by not more than 1. The possible arrangements of arrows for an arbitrary number r are shown in figure 1. If we assume that the case $r = 0$ corresponds to a left arrow, then figure 1 shows all possible vertex configurations with $r = 0, 1, 2, \ldots$. The six-vertex model results from the formulated model if one restricts the possible values of r to 0 and 1 in the cases (a), (b) and to the single value $r = 0$ in the cases (c), (d) in figure 1.

We will denote the Boltzmann weights of vertices by $w_a(r)$, $w_b(r)$, $w_c(r)$, $w_d(r)$. Putting all weights $w_\nu(r) = 1$ ($\nu = a, b, c, d$) one obtains a generalized ice model. If one considers the vertical coordinate of a lattice site as a discrete time, the model obtains a dynamical interpretation. Let $x_1, \ldots, x_n$ ($x_1 > x_2 > \ldots > x_n$) be the positions of the arrows pointing down in a row j. According to the formulated rules, the next row $j+1$ contains new positions of such arrows $x'_1, \ldots, x'_n$ obeying $x'_i \geq x_i$ for all $i = 1, 2, \ldots, n$. We will interpret an arrow pointing down in the row j as a particle at the moment j.
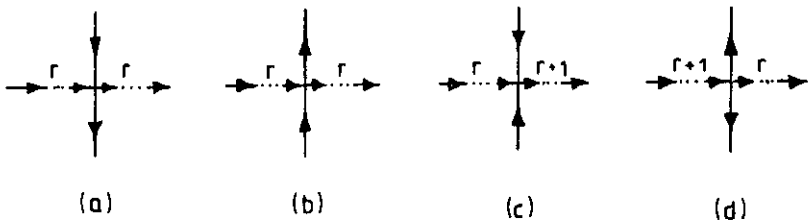

Figure 1. Arrow configurations allowed at a vertex.

Then the arrow arrangements on the lattice represent all possible one-side random walks of n hard-core particles. Such a dynamics differs from that defined by the six-vertex model. In the latter case all particles perform one-side random walks restricted by the condition that the coordinate of an ith particle at the moment $j+1$ does not exceed the coordinate of an $(i+1)$th particle at the moment j [1]. Our model permits all possible increments of particle coordinates obeying the single rule of no overtaking of each particle by a preceding one. So, we may call the introduced model a one-way traffic model.

Besides the arrow notation, one often uses the line representation by drawing a line along a bond instead of each right or down arrow. A typical state of a lattice row in this representation is shown in figure 2. We can see that there exists a line conservation law analogous to that for the six-vertex model. Ascribing the weight x to each line on a horizontal bond and the weight y to each vertical one, we get the Boltzmann weights of vertices shown in figure 1:

$$w_a(r) = x^r y \qquad w_b(r) = x^r \qquad w_c(r) = w_d(r) = x^{r+1/2} y^{1/2}. \qquad (2)$$

It can be seen that the model with the weights (2) belongs to a class of models with a different number of states on horizontal and vertical bonds, which was investigated in [19].
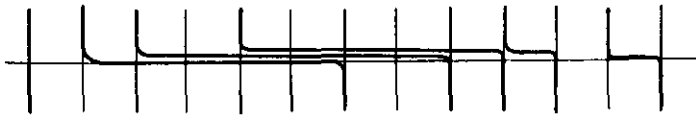

Figure 2. Typical arrangements of lines in adjacent rows.

Let us now draw attention to the boundary conditions. Firstly, consider the square lattice of M rows and N columns and assume periodic conditions in the horizontal direction. It is clear that for any arrangement of arrows in a row, fulfilling the generalized ice rule, there exists an arrangement which has R additional arrows on horizontal bonds, where R is an arbitrary positive integer. Therefore the number of arrow configurations per lattice site is infinite even for a finite number of columns. To make this number finite, it is enough to restrict the number of arrows r in some column of horizontal bonds, breaking the periodicity of the lattice.

The other possibility to get a finite number of configurations per site is to define periodic boundary conditions for the lattice drawn diagonally (figure 3). In this case, fixing the number of lines n in a cross-section of a cylinder, we get a finite number of arrow configurations in each row of vertices. The number of lines passing along a given bond does not exceed n. In the thermodynamic limit $N \to \infty$, $n \to \infty$, $n/N = \rho =$ constant, the number of possible states on a bond tends to infinity, and the entropy $S(\rho)$ becomes a continuous function of ρ. Obviously, $S(\rho)$ increases monotonically when $\rho \to \infty$.

Lastly we turn the lattice once more through $\pi/4$ and achieve the situation shown in figure 4. Here, as in the above case, one may investigate the ρ dependence of the entropy in the whole range $0 < \rho < \infty$.

In the next section we will obtain the Bethe ansatz equation for the diagonal boundary conditions. These equations can be generalized to an arbitrary angle of rotation by using the method suggested in [17].
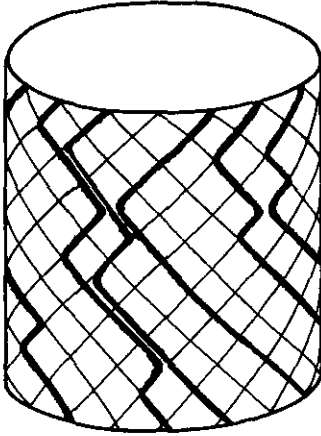
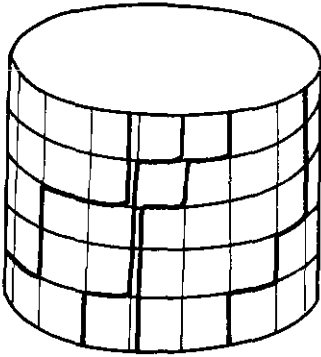**Figure 3.** Typical arrangements of lines on the lattice rotated through $\pi/4$.

Figure 4. The lattice rotated through $\pi/2$.

3. The method

The method which will be used to solve the model formulated in section 2 is based on the random walk representation of states of the system [17, 18]. In the case of models with a fixed number of components and for simple boundary conditions this method is completely equivalent to the transfer matrix technique. The random walk representation becomes useful when the number of permissible vertices grows infinitely with the lattice size and also when boundary conditions prevent the use of the translation-invariant transfer matrix.

We start rotating the lattice through $\pi/4$ and considering periodic boundary conditions. The lattice $\mathscr{L}$ consists of M rows, each of them containing N sites. To state the correspondence between vertex configurations and random walks, we consider a ring consisting of $2N$ points on which n random walks take place. The walks are supposed to start from lattice points, situated at even distances from each other, and to proceed as follows. At times $t = 1, 2, \ldots, M$ all walks make a step over one lattice spacing in one or another direction independently of the previous steps and independently of each other, except when two or more walkers have arrived at the same lattice point. If one draws the world lines of this set of walkers, plotting the position

horizontally and time vertically, one observes a similarity with the line representation of our model.

The trajectories of the random walks are restricted by the generalized ice rule: two or more walkers being at the same point can make simultaneous right steps but not left.

As usual, we begin an enumeration of possible configurations of the model with the case $n = 1$. This case corresponds obviously to a single unrestricted random walk of M steps. We will ascribe the weight $x\, e^{ik}$ to each right step and the weight $y\, e^{-ik}$ to each left one. Let us introduce the generating function of all t-stepped trajectories of a single random walk:

$$W_t(k) = (x\, e^{ik} + y\, e^{-ik})^t. \tag{3}$$

The coefficient of $e^{ik\Delta}$ in $W_t(k)$ is equal to a number of all possible walks starting at x_1 and ending at $x_1 + \Delta$ after t steps.

Consider now n random walks starting at the points $x_1, x_2, \ldots, x_n$. Our aim is to express the number of all possible trajectories of n restricted walks of M steps by the generating functions $W_M(k_1), \ldots, W_M(k_n)$ of n simple unrestricted random walks.

Following the Bethe ansatz prescription, we define a sum with certain coefficients $a(p)$:

$$Z_t = \sum_P a(p)\, W_t(k_{P(1)})\, W_t(k_{P(2)}) \ldots W_t(k_{P(n)}) \tag{4}$$

over $n!$ permutations of numbers $1, 2, \ldots, n$. The 'wavenumbers' $k_{P(1)}, k_{P(2)}, \ldots, k_{P(n)}$ correspond to random walks starting at $x_1, x_2, \ldots, x_n$ for each permutation P. The goal is to choose a complex-valued function $a(p)$ so that the contribution from all forbidden configurations of n walks cancel out. Then

$$Z_t = \sum_P a(p) F_t(k_{P(1)}, \ldots, k_{P(n)}) \tag{5}$$

where $F_t(k_{P(1)}, \ldots, k_{P(n)})$ is a weighted sum or a generating function of all configurations of n walks obeying the generalized ice rule. The walks described by F_t are labelled by $k_{P(i)}$ and ordered. The ordering means that for every two walks with initial positions x_i and x_j the difference of the increments $\Delta_i(t)$ and $\Delta_j(t)$ obeys $\Delta_i(t) - \Delta_j(t) \le (x_j - x_i)$ for all $t > 0$.

Let us now suppose that the set $\{k\}$ is such that

$$\sum_{i=1}^{n} k_i = 0 \tag{6}$$

and also assume periodic boundary conditions in both vertical and horizontal directions. Then each walk becomes closed. In this case

$$Z_M = F_M(0, 0, \ldots, 0) \sum_P a(p) \tag{7}$$

because the total weight of closed walks originating from the factors e^{ik_j} and e^{-ik_j} $(j = 1, 2, \ldots, n)$ is $\exp(\pm 2iNI(k_1 + k_2 + \ldots + k_n))$ where I is the number of synchronous rotations of the restricted walks around the torus in a given configuration. Due to (6) this factor equals unity.

The function on the right-hand side of (7) is just the one we want to find. By the definition of $F_M(k_1, \ldots, k_n)$ we have

$$F_M(0, 0, \ldots, 0) = \Lambda_{MN}(n) \tag{8}$$

where $\Lambda_{MN}(n)$ is the partition function of all allowed configurations of n walks on the $M \times N$ lattice with periodic boundary conditions.

The additional factor $\Sigma\, a(p)$ in (7) does not play any role in the limit of a very long cylinder or torus if $a(p)$ is a bounded function of $k_1, \ldots, k_n$ independent of M which yields

$$\lim_{M \to \infty} \left(\sum_P a(p) \right)^{t/M} = 1. \tag{9}$$

From (3), (4) and (8) we obtain the free energy per site:

$$f_n = -\frac{k_B T}{2MN} \ln \Lambda_{MN}(n) = -\frac{k_B T}{2N} \ln \Lambda_n \tag{10}$$

where

$$\Lambda_n = \lambda(k_1) \ldots \lambda(k_n) \tag{11}$$

with

$$\lambda(k) = x\, e^{ik} + y\, e^{-ik}. \tag{12}$$

Now, we have to find values of $k_1, \ldots, k_n$ which obey (6) and $a(p)$ for which the walk configurations violating the generalized ice rule will cancel out.

4. Bethe ansatz equations

Consider two walks with wavenumbers p and q passing through a certain site of $\mathscr{L}$. Let $\mathbb{P}$ and $\mathbb{Q}$ be two permutations of $\{k\}$ in which p and q exchange their positions: $\mathbb{P}\{k\} = \ldots p, q \ldots$ and $\mathbb{Q}\{k\} = \ldots q, p \ldots$. In figure 5 four possible continuations of walks corresponding to the permutation $\mathbb{P}$ are shown. For the permutation $\mathbb{Q}$ one must exchange the positions of p and q. The wavy lines denote a continuation of the unrestricted random walks.

According to definitions of the preceding section the weighted sum over all t-stepped walks from the given site in the case of figure $5(a)$ is

$$y^2\, e^{-ip-iq}\, W_{t-1}(p)\, W_{t-1}(q) \tag{13a}$$

and in the cases of figures $5(b)$, $5(c)$ and $5(d)$ respectively

$$xy\, e^{-iq+ip}\, W_{t-1}(p)\, W_{t-1}(q) \tag{13b}$$

$$xy\, e^{-ip+iq}\, W_{t-1}(p)\, W_{t-1}(q) \tag{13c}$$

$$x^2\, e^{ip+iq}\, W_{t-1}(p)\, W_{t-1}(q). \tag{13d}$$

The cases of figures $5(a)$ and $5(b)$ do not satisfy the rules of the model, so they must be eliminated.
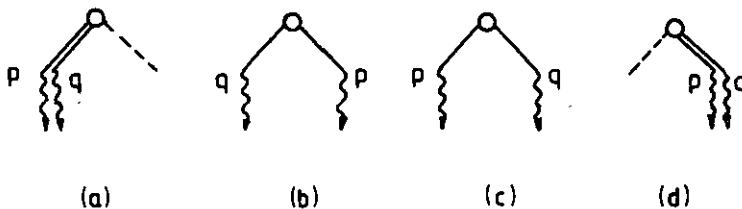

Figure 5. Four possible walk configurations at a site. Cases (a) and (b) must be eliminated.

Let $A(\mathbb{P})$ and $A(\mathbb{Q})$ be the weight factors of two walks at a given site, multiplied by the coefficients $a(\mathbb{P})$ and $a(\mathbb{Q})$. Using (13) we can write the cancellation conditions as follows:

$$A(\mathbb{P})(y^2\,\mathrm{e}^{-ip-iq} + xy\,\mathrm{e}^{ip-iq}) = -A(\mathbb{Q})(y^2\,\mathrm{e}^{-ip-iq} + xy\,\mathrm{e}^{-ip+iq}). \tag{14}$$

Defining the function $B(p, q)$ by the identity

$$A(\mathbb{P}) = B(p, q)A(\mathbb{Q}) \tag{15}$$

we get

$$B(p, q) = -\frac{\mathrm{e}^{-ip}(y\,\mathrm{e}^{-iq} + x\,\mathrm{e}^{iq})}{\mathrm{e}^{-iq}(y\,\mathrm{e}^{-ip} + x\,\mathrm{e}^{ip})} = -\frac{\mathrm{e}^{-ip}\lambda(q)}{\mathrm{e}^{-iq}\lambda(p)}. \tag{16}$$

Equations (14) and (15) provide the generalized ice rule condition for two-particle collisions. The essential property of the introduced model is the following: the permutation properties defined by (14)–(16) ensure the fulfilment of the generalized ice rule in the case of collision of more than two walkers at a given site. Indeed, let us consider m particles being at a given site at the same moment. A permutation $\mathbb{P}$ puts in correspondence to these particles the wavenumbers $k_{P(1)}, k_{P(2)}, \ldots, k_{P(m)}$. Due to conditions (14)–(16) applied to particles 1 and 2 only those continuations of the trajectories remain which fulfil the ice rule and for which particle 1 remains 'free', i.e. it has two possibilities to step from the given site while particle 2 can go only right. Using this fact, we consider the pair 1 and 3 independently of particle 2, and again we will obtain a 'free' motion of particle 1, and the deterministic step of particle 3. Proceeding further, we can prove the generalized ice rule for an arbitrary number of particles colliding at a site. Following the trajectories of walkers from the top row to the bottom one we delete from (4) all forbidden configurations.

Now consider the weight factors $A(\mathbb{P})$ and $A(\mathbb{Q})$ in more detail. Denote by x_0 and $x_0 + \delta$ the initial positions of two walks with wavenumbers p and q. Let x_c be a coordinate of a site where the walkers collide. Then the weight factor at the point x_c for the permutation $\mathbb{P}$ has the form

$$A(\mathbb{P}) = a(\mathbb{P})\,\mathrm{e}^{ip(x_c - x_0)}\,\mathrm{e}^{iq(x_c - x_0 - \delta)}\Omega(w_\xi) \tag{17a}$$

and for the permutation $\mathbb{Q}$

$$A(\mathbb{Q}) = a(\mathbb{Q})\,\mathrm{e}^{iq(x_c - x_0)}\,\mathrm{e}^{ip(x_c - x_0 - \delta)}\Omega(w_\xi) \tag{17b}$$

where $\Omega(w_\xi)$ is the Boltzmann weight of the walks originating from factors x or y ascribed to each right or left step, respectively. Substituting (17) into (15) we obtain

$$a(\mathbb{P}) = a(\mathbb{Q})B(p, q)\,\mathrm{e}^{i\delta(p-q)}. \tag{18}$$

Further, let $x_1^0, x_2^0, \ldots, x_n^0$ be the initial positions of the walkers with wavenumbers $k_1, \ldots, k_n$ in the permutation $\mathbb{P}$. Denote the difference $x_{j+1}^0 - x_j^0$ by $\delta_{j,j+1}$. The cyclic permutation $\mathbb{Q} = \{2, 3, \ldots, n, 1\}$ gives the coefficient $a(\mathbb{Q})$ associated with $a(\mathbb{P})$ by

$$a(\mathbb{P}) = a(\mathbb{Q})\prod_{j=2}^{n} B(k_1, k_j)\,\mathrm{e}^{i\delta_{12}(k_2 - k_1)}\ldots\mathrm{e}^{i\delta_{n-1,n}(k_n - k_1)}. \tag{19}$$

On the other hand the relative positions of the walkers in $\mathbb{P}$ and $\mathbb{Q}$ coincide. So, we must satisfy the identity $a(\mathbb{P}) = a(\mathbb{Q})$. This condition gives

$$\prod_{j=2}^{n} B(k_1, k_j) = e^{i\delta_{12}(k_1 - k_n)} \ldots e^{i\delta_{n-1,n}(k_1 - k_n)}. \tag{20}$$

Noting that

$$\sum_{j=1}^{n} \delta_{j,j+1} = 2N \tag{21}$$

one can represent the right-hand side of (20) as

$$e^{2ik_1 N} D(\{k\})$$

where

$$D(\{k\}) = \prod_{j=1}^{n} e^{-i\delta_{j,j+1} k_j}. \tag{22}$$

As the thermodynamic properties of the model in the limit of an infinitely long cylinder ($M \to \infty$) do not depend on the initial positions of particles, we can put all $\delta_{j,j+1}$ equal to each other. Note that it is not necessary to associated the choice of $\delta_{j,j+1}$ with the real initial positions of walks because one can always ascribe an arbitrary initial factor to the weight of a walk.

As mentioned above, we must find the set $\{k\}$ which obeys (6). Then from the identity of all $\delta_{j,j+1}$, it follows that

$$D(\{k\}) = 1. \tag{23}$$

Using (22) and (23) we may rewrite (20) in the standard form

$$e^{2ik_j N} = - \prod_{i=1}^{n} B(k_j, k_i). \tag{24}$$

As usual, we have n equations for $k_1, k_2, \ldots, k_n$. These can be solved and then (10) and (11) give the free energy density for our model.

In this paper we shall not consider the solution of (24). Instead, we shall find a complete solution of the problem for a finite square lattice with the boundary conditions described in [14].

5. Finite lattice

Consider a square of size $N \times N$ and define the boundary conditions as follows. Put an arrow pointing right ($r = 1$) on each horizontal bond in the left-most column. Further, let each horizontal bond in the right-most column be provided with an arrow pointing left ($r = 0$). Also put an arrow up on each vertical bond in the upper row and an arrow down on each vertical bond in the lower row. In the line representation, we get typical configurations such as the one shown in figure 6 for the case $N = 6$. The problem consists in the enumeration of all possible line configurations T_N for an arbitrary N.
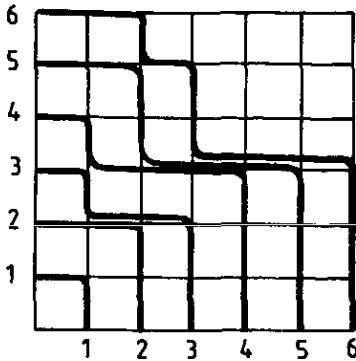
Figure 6. Korepin's boundary conditions for the square.

We begin with the consideration of an auxiliary problem on a semi-infinite strip of width N. The new boundary conditions are clear from figure 7. They coincide with the preceding case with the exception of vertical bonds in the lower row. No direction of arrows on these bonds is fixed.

Consider directed random walks on the strip starting at the points $(1, 1)$, $(1, 2), \ldots, (1, N)$ and ending at the lower boundary. The generating function of walks starting at the point $(1, n)$ has the form

$$W_n(k) = y^n \lambda^n(k) \tag{25}$$

where

$$\lambda(k) = \frac{1}{1 - x\,e^{ik}} \tag{26}$$

is the generating function of all possible motions in a given row. As the number of vertical bonds is the same for all configurations, we can choose $y = 1$ without loss of generality.

Following the method outlined in section 4, we write a sum over permutations of the number $1, 2, \ldots, N$ similar to (4):

$$Z_N = \sum_P a(p) W_1(k_{P(1)}) \ldots W_N(k_{P(N)}). \tag{27}$$
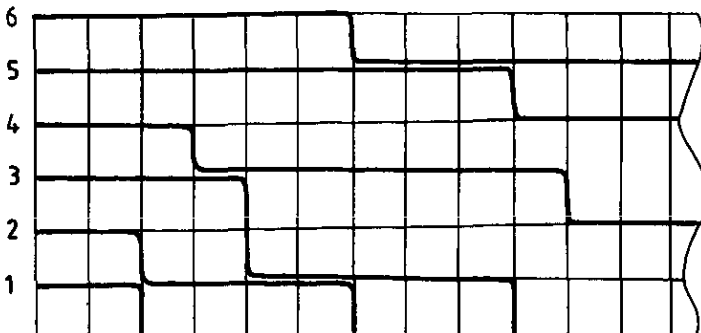

Figure 7. The auxiliary model on the semi-infinite strip.

The cancellation condition for forbidden configurations can be found as in section 4. For a pair of permutations $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{P}\{k\} = \ldots p, q \ldots$ and $\mathbb{Q}\{k\} = \ldots q, p \ldots$ we have similarly to (16)

$$B(p, q) = -\frac{A(\mathbb{P})}{A(\mathbb{Q})} = -\frac{\lambda(q)}{\lambda(p)}. \tag{28}$$

Note that two walkers colliding at a point with the horizontal coordinate x have the same factor $e^{ik(p-q)}$ for both permutations $\mathbb{P}$ and $\mathbb{Q}$. Therefore, by the definition of the coefficients $a(\mathbb{P})$ and $A(\mathbb{P})$ it follows that

$$\frac{a(\mathbb{P})}{a(\mathbb{Q})} = \frac{A(\mathbb{P})}{A(\mathbb{Q})}. \tag{29}$$

We have shown above that fulfilment of the generalized ice rules for two particles guarantees their fulfilment for any number of particles colliding at a lattice site. The factorization of $B(p, q)$ given by (28) plays an essential role in the derivation of a closed expression for the partition function.

It is easy to check that the coefficients $a(p)$ obeying (28) and (29) must have the form

$$a(p) = \delta_p \lambda(k_{P(1)}) \lambda^2(k_{P(2)}) \ldots \lambda^N(k_{P(N)}) \tag{30}$$

where $\delta_p = 1$ for even permutations and $\delta_P = -1$ for odd ones.

Inserting (30) into (27) we get

$$Z_N = \sum_P \delta_p \lambda^2(k_{P(1)}) \lambda^4(k_{P(2)}) \ldots \lambda^{2N}(k_{P(N)}). \tag{31}$$

On the other hand, due to cancellation of forbidden configurations we have

$$Z_N = \sum_P a(p) F(k_{P(1)}, \ldots, k_{P(N)}) \tag{32}$$

where F is the generating function of N random walks on the strip restricted by the collision rules. The desired number of line configurations on the square T_N is determined by the first term in an expansion of the function in the left-hand side of (32):

$$Z_N = T_N x^{N(N-1)/2} \sum_P \delta_P \prod_{j=1}^N e^{ik_{P(j)}(j-1)} + O(x^{N(N-1)/2+1}). \tag{33}$$

To find T_N, we rewrite (31) and (33) using the Vandermond determinants:

$$Z_N = \prod_{j=1}^N \lambda^{2N}(k_j) \det M_1 \tag{34}$$

where

$$M_1 = \begin{Vmatrix} 1 & (1-x\,e^{ik_1})^2 & (1-x\,e^{ik_1})^4 & \ldots & (1-x\,e^{ik_1})^{2(N-1)} \\ 1 & (1-x\,e^{ik_2})^2 & (1-x\,e^{ik_2})^4 & \ldots & (1-x\,e^{ik_2})^{2(N-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & (1-x\,e^{ik_N})^2 & (1-x\,e^{ik_N})^4 & \ldots & (1-x\,e^{ik_N})^{2(N-1)} \end{Vmatrix} \tag{35}$$

and

$$Z_N = T_N x^{N(N-1)/2} \det M_2 + O(x^{N(N-1)/2+1}) \tag{36}$$

where

$$
M_2 = \begin{Vmatrix}
1 & e^{ik_1} & e^{2ik_1} & \ldots & e^{(N-1)ik_1} \\
1 & e^{ik_2} & e^{2ik_2} & \ldots & e^{(N-1)ik_2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
1 & e^{ik_N} & e^{2ik_N} & \ldots & e^{(N-1)ik_N}
\end{Vmatrix}. \tag{37}
$$

Evaluating the determinant in (34) we obtain

$$
\det M_1 = \prod_{m>n} [(1 - x\,e^{ik_m})^2 - (1 - x\,e^{ik_n})^2]
$$

$$
= x^{N(N-1)/2} \prod_{m>n} [(2 - x\,e^{ik_m} - x\,e^{ik_n})(e^{ik_m} - e^{ik_n})]. \tag{38}
$$

Equation (38) gives the first term of the x-expansion:

$$
\det M_1 = x^{N(N-1)/2} 2^{N(N-1)/2} \det M_2 + O(x^{N(N-1)/2+1}). \tag{39}
$$

Taking into account that

$$
\lim_{x\to 0} \lambda(k_j) = 1 \qquad j = 1, \ldots, N \tag{40}
$$

and comparing (36) and (38) we finally get

$$
T_N = 2^{N(N-1)/2}. \tag{41}
$$

There is another way to extract T_N from the comparison of (31) and (33). Let us choose a term in the sum of (33), for example

$$
\exp[i(k_2 + 2k_3 + 3k_4 + \ldots + (N-1)k_N)]
$$

and find the coefficient of this term in (31). To this goal we write (31) in the form

$$
Z_N = \prod_{j=1}^{N} \lambda^{2N}(k_j) \sum_P \delta_p (1 - x\,e^{ik_{P(1)}})^{2(N-1)} \ldots (1 - x\,e^{ik_{P(N-1)}})^2. \tag{42}
$$

According to (32), all terms in the latter sum of an order less than $x^{N(N-1)/2}$ cancel out. Then the contribution to the desired coefficient from a permutation p is

$$
\delta_p \binom{2(N-1)}{P(1)} \binom{2(N-2)}{P(2)} \ldots \binom{2}{P(N-1)} \tag{43}
$$

where

$$
\binom{n}{m}
$$

is the binomial coefficient. We put, as usual,

$$
\binom{n}{m} = 0
$$

if $m > n$.

The total contribution to T_N from all permutations equals the determinant of the matrix $a(i, j)$:

$$
a(i, j) = \binom{2j}{i}. \tag{44}
$$

Comparing the obtained result with (41) we get an interesting formula†

$$T_N = \det \begin{vmatrix} \binom{2}{1} & \binom{4}{1} & \cdots & \binom{2N-2}{1} \\ \binom{2}{2} & \binom{4}{2} & \cdots & \binom{2N-2}{2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \binom{2}{N-1} & \binom{4}{N-1} & \cdots & \binom{2N-2}{N-1} \end{vmatrix} = 2^{N(N-1)/2}. \tag{45}$$

5. Discussion

The molecular freedom per site defined by

$$W = \lim_{N \to \infty} T_N^{1/N^2} \tag{46}$$

is $W = \sqrt{2}$ due to (41). This result permits us to obtain an upper bound for the molecular freedom of the ice model W_{ice} with Korepin's boundary conditions [14]. Indeed, the number of states of the six-vertex model does not exceed the one for the considered model. Therefore we have

$$W_{ice} \leqslant \sqrt{2} < (\tfrac{4}{3})^{3/2}. \tag{47}$$

The latter quantity is the known result of Lieb [1] for the ice model with periodic boundary conditions.

It is interesting to note that $\ln T_N$ given by (41) is the sum of a bulk term which is proportional to N^2 and a surface one proportional to N. Thus all finite-size corrections are exactly equals to zero.

For models which possess conformal invariance the $1/N^2$-order corrections to the free energy are governed by the central charge of the Virasoro algebra. The absence of correction terms means, probably, that the model with Korepin's boundary conditions is out of criticality.

The examination of conformal properties of the presented model is of considerable interest because the form of the 'scattering function' $B(p, q)$ indicates that the model takes on an intermediate place between the free and interaction fermion models. Indeed, $B(p, q)$ depends on p and q but is factorizable. It is interesting to check whether the factorization leads to the free fermion value of central charge $C = \tfrac{1}{2}$.

---

† Similar combinatorical arguments lead to the more general formula

$$\det \begin{vmatrix} \binom{k}{1} & \binom{2k}{1} & \cdots & \binom{nk}{1} \\ \binom{k}{2} & \binom{2k}{2} & \cdots & \binom{nk}{2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \binom{k}{n} & \binom{2k}{n} & \cdots & \binom{nk}{n} \end{vmatrix} = k^{n(n+1)/2}.$$

Thermodynamic properties of the model with periodic boundary conditions obviously depend on the activities x and y. In particular, it is easy to find a line of phase transitions bounding the ordered phase. Similarly to the six-vertex model, the critical values of x and y can be found from the consideration of one-particle excitations of the ground state ($n = 1$). Writing the generating function (3) of a complex variable z,

$$W_t(z) = \lambda^t(z) = (xz + yz^{-1})^t \tag{48}$$

we get x_c and y_c from the condition $|z_0| < 1$ where z_0 is a root minimal in absolute value of the equation $\lambda(z) = 1$. Finding z_0 we get $x_c + y_c = 1$. The form of the singularity at the critical point needs a detailed investigation of the Bethe ansatz equations (16) and (24).

Acknowledgments

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